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Inequalities of trapezoidal type involving generalized fractional integrals

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Abstract During the last years several fractional integrals were investigated. Having this idea in mind, in the present article, some new generalized fractional integral inequalities of the trapezoidal type for λ_φ–preinvex functions, which are differentiable and twice differentiable, are established. Then, by employing those results, we explore the new estimates on trapezoidal type inequalities for classical integral and Riemann–Liouville fractional integrals, respectively. Finally, we apply our new inequalities to construct inequalities involving moments of a continuous random variable.

1. Introduction

Convexity has been received considerable attention due to its applications in various fields of applied and pure sciences; for example engineering, management science, economics, finance and optimization theory. Convex functions have been generalized and extended in optimization, mathematical inequalities and several directions using various innovative techniques [9,23,24,26, 28,34]. The most significant generalization of the convex function is the invex function which was introduced by Hanson [12]. After that, Weir and Mond [42] introduced the concept of preinvex function. Indeed, preinvex functions are playing significant role to the study of equilibrium problems, optimization theory, integral inequalities, fractional integral inequalities and so forth. Furthermore, preinvex function and its basic properties were studied by many researchers; for more details see [2,19,20,27,28]. Typical applications of the classical inequalities are: probabilistic problems, decision making in structural engineering and fatigue life. Some new results can be seen in the following references [41–47].

Now, we recall the relevant definitions and preliminary results concerning preinvexity and trapezoidal type inequalities.

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Definition 1.1. [33] Let $H \subseteq \mathbb{R}^n$ and $\eta : H \times H \to \mathbb{R}^n$. Then the set H is said to be invex with respect to η, if

$$w + \zeta\eta(y, w) \in H \tag{1.1}$$

for every $\zeta \in [0, 1]$ and $w, y \in H$. An invex set H associated with function η is also called η-connected set.

Remark 1.1. Particularly, if function η is specialized by $\eta(y, w) = y - w$ for all $y, w \in \mathbb{R}^n$, then Definition 1.1 reduces to the definition of convexity.

Definition 1.2. [33] Let $H \subseteq \mathbb{R}^n$ and $\eta : H \times H \to \mathbb{R}^n$. We say that a function $g : H \to \mathbb{R}$ is preinvex with respect to η on H, if

$$g(w + \zeta\eta(y, w)) \leqslant (1 - \zeta)g(w) + \zeta\, g(y) \tag{1.2}$$

for every $\zeta \in [0, 1]$ and $w, y \in H$. However, if the function $-g$ is preinvex, then we say that g is preincave.

Definition 1.3. [11] Let $H \subseteq \mathbb{R}^n, \eta : H \times H \to \mathbb{R}^n$ and $\varphi : H \to \mathbb{R}$. Then, the set H is said to be φ-invex with respect to η and φ, if

$$w + \zeta e^{i\varphi}\eta(y, w) \in H \tag{1.3}$$

for every $\zeta \in [0, 1]$ and $w, y \in H$.

Remark 1.2. Especially, if we select

(i) $\varphi = 0$, then φ-invexity reduces to the usual invexity.
(ii) $\eta(y, w) = y - w$, then Definition 1.3 reduces to the definition of φ-convexity.
(iii) $\varphi = 0$ and $\eta(y, w) = y - w$, then Definition 1.3 reduces to th definition of convexity.

Definition 1.4. [8] Let $H \subseteq \mathbb{R}^n$ and $\eta : H \times H \to \mathbb{R}^n$. A nonnegative function $g : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be λ_φ-preinvex on H with respect to φ and η, if

$$g\big(w + \zeta e^{i\varphi}\eta(y, w)\big) \leqslant \frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}g(y) + \frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}g(w) \tag{1.4}$$

for every $\zeta \in [0, 1]$, $w, y \in I$ and $\varphi \in \left[0, \frac{\pi}{2}\right]$.

Definition 1.5. [[30]] Let $g : [u, v] \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on an open interval (u, v) with the second derivative bounded on the interval (u, v); that is, $\|g''\|_\infty := \sup_{x\in(u,v)}|g''(x)| < \infty$, then the trapezoidal type inequality are defined by:

$$\left|\int_u^v g(x)dx - \frac{v-u}{2}[g(u) + g(v)]\right| \leqslant \frac{(v-u)^3}{12}\|g''\|_\infty, \tag{1.5}$$

From a complementary viewpoint to Ostrowski type inequalities [10] and mid-point type inequalities [9], trapezoidal type inequality provides a priory error bounds in approximating the Riemann integralby a (generalized) trapezoidal formula [30]. Just like in the case of Ostrowski's inequality the development of these kind of results have registered a sharp growth in the last decade with more than 50 papers published, as one can easily asses this by performing a search with the key word trapezoid and inequality in the title of the papers reviewed by MathSciNet database of the American Mathematical Society. Numerous extensions, generalizations in both the integral and discrete case have been discovered (see [9,36]). More general versions form-time differentiable functions, the corresponding versions on time scales, for vector valued functions or multiple integrals have been established as well (see [6,29]). Numerous applications in Numerical Analysis, Probability Theory, and other fields have been also given.

In [8], Ermeydan and Yildirim established the following trapezoidal type equality and inequalities for differential λ_φ-preinvex functions:

Lemma 1.1. *Let $g : [u, v] \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on an open interval (u, v) with $\eta(v, u) > 0, u < v$. If $g' \in L[u, u + e^{i\varphi}\eta(v, u)]$, then*

$$\begin{aligned}&\frac{g(u) + g(u + e^{i\varphi}\eta(v, u))}{2} \\ &\quad - \frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\left[J^\mu_{u^+}g\big(u + e^{i\varphi}\eta(v, u)\big) + J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right] \\ &\quad = \frac{e^{i\varphi}\eta(v, u)}{2}\int_0^1 [(1-\zeta)^\mu - \zeta^\mu]g'\big(u + (1-\zeta)e^{i\varphi}\eta(v, u)\big)d\zeta.\end{aligned} \tag{1.6}$$

Theorem 1.1. *Let $g : [u, v] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $|g'| \in L[u, u + e^{i\varphi}\eta(v, u)]$ is decreasing, measurable and λ_φ-preinvex function on a closed interval $[u, v]$ for $\eta(v, u) > 0$ with $0 \leqslant u < v$ and $\mu > 0$, then*

$$\begin{aligned}&\Bigg|\frac{g(u) + g(u + e^{i\varphi}\eta(v, u))}{2} - \frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu} \\ &\left[J^\mu_{u^+}g\big(u + e^{i\varphi}\eta(v, u)\big) + J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]\Bigg| \\ &\quad \leqslant \frac{e^{i\varphi}\eta(v, u)}{4}\left(|g'(u)| + \frac{1-\lambda}{\lambda}|g'(v)|\right) \\ &\quad \times \left[\beta_{\frac{1}{2}}\left(\frac{1}{2}, \mu + \frac{1}{2}\right) - \beta_{\frac{1}{2}}\left(\mu + \frac{1}{2}, \frac{1}{2}\right)\right],\end{aligned} \tag{1.7}$$

where $\beta_z(u, v)$ is the incomplete beta function (see, e.g. [5])

$$\beta_z(u, v) = \int_0^z \zeta^{u-1}(1-\zeta)^{v-1}d\zeta.$$

Theorem 1.2. *Let $g : [u, v] \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $|g'|^q \in L[u, u + e^{i\varphi}\eta(v, u)]$ is decreasing, measurable and λ_φ-preinvex function on a closed interval $[u, v]$ for $\eta(v, u) > 0$ with $0 \leqslant u < v$ and $\mu > 0$, then*

$$\begin{aligned}&\left|\frac{g(u) + g(u + e^{i\varphi}\eta(v, u))}{2} - \frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\left[J^\mu_{u^+}g\big(u + e^{i\varphi}\eta(v, u)\big)\right.\right. \\ &\left.\left.+J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]\right| \leqslant \frac{e^{i\varphi}\eta(v, u)}{2}\left\{\frac{\pi}{4}|g'(u)|^q + \frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|g'(v)|^q\right\}^{\frac{1}{q}}\left(\frac{2-2^{1-\mu p}}{\mu p+1}\right)^{\frac{1}{p}},\end{aligned} \tag{1.8}$$

where $\mu > 0, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function. If $|g'|^q\in L[u,u+e^{i\varphi}\eta(v,u)]$ for some $q\geqslant 1$ is decreasing, measurable and λ_φ- preinvex function on a closed interval $[u,v]$ for $\eta(v,u)>0$ with $0\leqslant u<v$ and $\mu>0$, then*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}-\frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\left[J^\mu_{u^+}g\left(u+e^{i\varphi}\eta(v,u)\right)\right.\right.$$
$$\left.\left.+J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]\right|\leqslant\frac{e^{i\varphi}\eta(v,u)}{2^{\frac{1}{q}}}\left(\frac{1-2^{-\mu}}{\mu+1}\right)^{\frac{q-1}{q}}\left(\frac{|g'(u)|^q}{2}\left[\beta_{\frac12}\left(\frac12,\mu+\frac12\right)\right.\right.$$
$$\left.\left.-\beta_{\frac12}\left(\mu+\frac12,\frac12\right)\right]+\left(\frac{1-\lambda}{\lambda}\right)\frac{|g'(v)|^q}{2}\left[\beta_{\frac12}\left(\frac12,\mu+\frac12\right)-\beta_{\frac12}\left(\mu+\frac12,\frac12\right)\right]\right)^{\frac1q}. \tag{1.9}$$

After that, they also proved the following trapezoidal type equality and inequalities for twice differential λ_φ-preinvex functions.

Lemma 1.2. Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on an open interval (u,v) with $\eta(v,u)>0, u<v$. If $g''\in L[u,u+e^{i\varphi}\eta(v,u)]$, then

$$\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}$$
$$-\frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\left[J^\mu_{u^+}g\left(u+e^{i\varphi}\eta(v,u)\right)+J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]$$
$$=\frac{(e^{i\varphi}\eta(v,u))^2}{2(\mu+1)}\int_0^1\left[1-(1-\zeta)^{\mu+1}-\zeta^{\mu+1}\right]g''$$
$$\times\left(u+(1-\zeta)e^{i\varphi}\eta(v,u)\right)d\zeta. \tag{1.10}$$

Theorem 1.4. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function. If $|g''|\in L[u,u+e^{i\varphi}\eta(v,u)]$ is decreasing, measurable and λ_φ-preinvex function on a closed interval $[u,v]$ for $\eta(v,u)>0$ with $0\leqslant u<v$ and $\mu>0$, then*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}-\frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\left[J^\mu_{u^+}g\left(u+e^{i\varphi}\eta(v,u)\right)\right.\right.$$
$$\left.\left.+J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]\right|\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{4(\mu+1)}\left\{|g''(u)|\left[\frac{\pi}{2}-\beta\left(\frac32,\mu+\frac32\right)\right.\right.$$
$$\left.\left.-\beta\left(\mu+\frac52,\frac12\right)\right]|g''(v)|\left[\frac{\pi}{2}-\beta\left(\frac12,\mu+\frac52\right)-\beta\left(\mu+\frac32,\frac32\right)\right]\right\}, \tag{1.11}$$

where $\beta(u,v)$ is the well-known beta function given by

$$\beta(u,v)=\int_0^1\zeta^{u-1}(1-\zeta)^{v-1}d\zeta.$$

Theorem 1.5. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function. If $|g''|^q\in L[u,u+e^{i\varphi}\eta(v,u)]$ is decreasing, measurable and λ_φ-preinvex function on a closed interval $[u,v]$ for $\eta(v,u)>0$ wit $0\leqslant u<v$ and $\mu>0$, then*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}-\frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\right.$$
$$\left.\left[J^\mu_{u^+}g\left(u+e^{i\varphi}\eta(v,u)\right)+J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]\right|$$
$$\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{2(\mu+1)}\left(1-2^{1-\mu}\right)\left\{\frac{\pi}{4}|g''(u)|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|g''(v)|^q\right\}^{\frac1q}, \tag{1.12}$$

where $\mu>0, q>1$ and $\frac1p+\frac1q=1$.

Theorem 1.6. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function. If $|g''|^q\in L[u,u+e^{i\varphi}\eta(v,u)]$ for some $q\geqslant1$ is decreasing, measurable and λ_φ- preinvex function on a closed interval $[u,v]$ for $\eta(v,u)>0$ with $0\leqslant u<v$ and $\mu>0$, then*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}-\frac{\Gamma(\mu+1))}{2(e^{i\varphi}\eta(v,u))^\mu}\left[J^\mu_{u^+}g\left(u+e^{i\varphi}\eta(v,u)\right)\right.\right.$$
$$\left.\left.+J^\mu_{(u+e^{i\varphi}\eta(v,u))^-}g(u)\right]\right|\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{2(\mu+1)}\left(1-2^{1-\mu}\right)^{\frac{q-1}{q}}$$
$$\times\left(\frac{|g''(u)|^q}{2}\left[\frac{\pi}{2}-\beta\left(\frac32,\mu+\frac32\right)-\beta\left(\mu+\frac52,\frac12\right)\right]\right.$$
$$\left.+\frac{|g''(v)|^q}{2}\left[\frac{\pi}{2}-\beta\left(\frac12,\mu+\frac52\right)-\beta\left(\mu+\frac32,\frac32\right)\right]\right)^{\frac1q}. \tag{1.13}$$

Besides, we shall recall some useful definitions and mathematical preliminaries for the generalized fractional integral operators [30]:

Let $\rho:[0,\infty)\to[0,\infty)$ be such that

$$\int_0^1\frac{\rho(\zeta)}{\zeta}d\zeta<\infty.$$

The left- and right-sided generalized fractional integral operators are defined as follows:

$${}_{u^+}I_\rho g(x)=\int_u^x\frac{\rho(x-\zeta)}{x-\zeta}g(\zeta)d\zeta,\quad x>u, \tag{1.14}$$
$${}_{v^-}I_\rho g(x)=\int_x^v\frac{\rho(\zeta-x)}{\zeta-x}g(\zeta)d\zeta,\quad x<v, \tag{1.15}$$

respectively. For more details; see [27,36].

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral [32,37], k-Riemann-Liouville fractional integral [26], Katugampola fractional integrals [13–15], conformable fractional integral [22,40] [25], Hadamard fractional integrals [32,43], and so on. More specifically,

(i) if $\rho(\zeta)=\zeta$, then (1.14) and (1.15) reduce to the classical Riemann integrals:

$I_{u^+}g(x)=\int_u^x g(\zeta)d\zeta,\quad x>u,$
$I_{v^-}g(x)=\int_x^v g(\zeta)d\zeta,\quad x<v.$

(ii) when ρ is specialized by $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, then (1.14) and (1.15) become the Riemann-Liouville fractional integrals:

$I^\mu_{u^+}g(x)=\frac{1}{\Gamma(\mu)}\int_u^x(x-t)^{\mu-1}g(\zeta)d\zeta,\quad x>u,$
$I^\mu_{v^-}g(x)=\frac{1}{\Gamma(\mu)}\int_x^v(\zeta-x)^{\mu-1}g(\zeta)d\zeta,\quad x<v.$

(iii) while ρ is taken by $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then (1.14) and (1.15) transfer to the k-Riemann–Liouville fractional integrals:

$I^{\mu}_{u^+,k}g(x) = \frac{1}{k\Gamma_k(\mu)}\int_u^x (x-t)^{\frac{\mu}{k}-1}g(\zeta)d\zeta, \quad x > u,$

$I^{\mu}_{v^-,k}g(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^v (\zeta-x)^{\frac{\mu}{k}-1}g(\zeta)d\zeta, \quad x < v,$

where $\Gamma_k(\mu)$ is defined by (see cf. Mubeen and Habibullah [32])

$$\Gamma_k(\mu) = \int_0^{\infty} \zeta^{\mu-1}e^{-\frac{\zeta^k}{k}}d\zeta, \quad \mathbb{R}(\mu) > 0$$

with

$$\Gamma_k(\mu) = k^{\frac{\mu}{k}-1}\Gamma\left(\frac{\mu}{k}\right), \quad \mathbb{R}(\mu) > 0;\ k > 0.$$

2. Trapezoidal type inequalities for differential functions

This section is devoted to establish some new trapezoidal type inequalities for λ_{φ}–preinvex functions, which are differentiable or twice differentiable.

To obtain the trapezoidal type inequalities for differential λ_{φ}–preinvex functions, we need the following lemma.

Lemma 2.1. *Let $g : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on an open interval (u,v) such that $g' \in L[u, u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u) > 0$ and $u < v$. Then*

$$\begin{aligned}&\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_{\rho}g\big(u+e^{i\varphi}\eta(v,u)\big) + {}_{(u+e^{i\varphi}\eta(v,u))^-}I_{\rho}g(u)\right]\\ &= \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left[\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)]g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\right], \end{aligned} \tag{2.1}$$

where

$$\Upsilon(\zeta) = \int_0^{\zeta} \frac{\rho(e^{i\varphi}\eta(v,u)x)}{x}dx < \infty. \tag{2.2}$$

Proof. Using integrating by parts, it yields

$$\begin{aligned}J &= \int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)]g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\\ &= \frac{1}{e^{i\varphi}\eta(v,u)}\Bigg[\big\{g(u)+g\big(u+e^{i\varphi}\eta(v,u)\big)\big\}\Upsilon(1)\\ &\quad + \int_0^1 \frac{\rho(e^{i\varphi}\eta(v,u)(1-\zeta))}{1-\zeta}g\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\\ &\quad + \int_0^1 \frac{\rho(e^{i\varphi}\eta(v,u)\zeta)}{\zeta}g\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\Bigg].\end{aligned}$$

Then, by changing the variable x by $x = u+(1-\zeta)e^{i\varphi}\eta(v,u)$, we get

$$\begin{aligned}J &= \frac{1}{e^{i\varphi}\eta(v,u)}\Bigg[\big\{g(u)+g\big(u+e^{i\varphi}\eta(v,u)\big)\big\}\Upsilon(1) - \int_u^{u+e^{i\varphi}\eta(v,u)} \frac{\rho(x-u)}{x-u}g(x)dx\\ &\quad - \int_u^{u+e^{i\varphi}\eta(v,u)} \frac{\rho(u+e^{i\varphi}\eta(v,u)-x)}{u+e^{i\varphi}\eta(v,u)-x}g(x)dx\Bigg]\\ &= \frac{1}{e^{i\varphi}\eta(v,u)}\Big[\big\{g(u)+g\big(u+e^{i\varphi}\eta(v,u)\big)\big\}\Upsilon(1)\\ &\quad - \big({}_{(u+e^{i\varphi}\eta(v,u))-}I_{\rho}g(u) + {}_{u+}I_{\rho}g\big(u+e^{i\varphi}\eta(v,u)\big)\big)\Big]. \end{aligned} \tag{2.3}$$

Therefore, it is not difficult to rearrange the above equality to the desired result (2.1). □

Remark 2.1. It is obviously that if ρ is specialized by $\rho(\zeta) = \frac{\zeta^{\mu}}{\Gamma(\mu)}$, then identity (2.1) reduces to the one (1.6).

Additionally, if $\rho(\zeta) = \frac{\zeta^{\mu}}{\Gamma(\mu)}$, $\eta(v,u) = v-u$ and $\varphi = 0$, then (2.1) becomes to the following identity

$$\begin{aligned}&\frac{g(u)+g(v)}{2} - \frac{\Gamma(\mu+1))}{2(v-u)^{\mu}}\left[I^{\mu}_{u^+}g(v) + I^{\mu}_{v^-}g(u)\right]\\ &= \frac{v-u}{2}\int_0^1 [(1-\zeta)^{\mu}-\zeta^{\mu}]g'(\zeta u+(1-\zeta)v)d\zeta,\end{aligned}$$

which was obtained by Sarikaya et al. [36].

Moreover, if $\rho(\zeta) = \frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then for k-Riemann–Liouville fractional integral, we have

$$\begin{aligned}&\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\\ &- \frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^{\mu}_{u^+,k}g\big(u+e^{i\varphi}\eta(v,u)\big) + I^{\mu}_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\\ &= \frac{(e^{i\varphi}\eta(v,u))^2}{2(\mu+k)}\\ &\quad \times \int_0^1 \left[(1-\zeta)^{\frac{\mu}{k}+1}-\zeta^{\frac{\mu}{k}+1}\right]g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta.\end{aligned}$$

Theorem 2.1. *Let $g : [u,v] \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on an open interval (u,v) such that $g' \in L[u, u+e^{i\varphi}\eta(v,u)]$ with $u < v$ and $\eta(v,u) > 0$. If $|g'|$ is decreasing, measurable and λ_{φ}-preinvex on a closed interval $[u,v]$, then it holds*

$$\begin{aligned}&\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_{\rho}g\big(u+e^{i\varphi}\eta(v,u)\big)\right.\right.\\ &\left.\left.+{}_{(u+e^{i\varphi}\eta(v,u))^-}I_{\rho}g(u)\right]\right| \leqslant \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}(A|g'(u)|+B|g'(v)|), \end{aligned} \tag{2.4}$$

where the constants A, and B are given by

$A = \int_0^1 \frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}|\Upsilon(1-\zeta)-\Upsilon(\zeta)|d\zeta,$

$B = \int_0^1 \frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}|\Upsilon(1-\zeta)-\Upsilon(\zeta)|d\zeta.$

Proof. Using equality (2.1) and the property of modulus, we have

$$\begin{aligned}&\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_{\rho}g\big(u+e^{i\varphi}\eta(v,u)\big) + {}_{(u+e^{i\varphi}\eta(v,u))^-}I_{\rho}g(u)\right]\right|\\ &= \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left|\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)]g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\right|\\ &\leqslant \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\int_0^1 |\Upsilon(1-\zeta)-\Upsilon(\zeta)|\left|g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\right|d\zeta.\end{aligned}$$

Recall that $|g'|$ is λ_{φ}-preinvex function on a closed interval $[u,v]$, so, for any $\zeta \in (0,1)$ it finds

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u))\right.\right.$$
$$\left.\left.+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right| \leqslant \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}$$
$$\times\left[\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\left(\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}|g'(u)|d\zeta+\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}|g'(v)|\right)\right]$$
$$\leqslant\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left[|g'(u)|\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}d\zeta+|g'(v)|\right.$$
$$\left.\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}\right]=\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}(A|g'(u)|+B|g'(v)|).$$

This directly implies the desired inequality (2.4). □

Remark 2.2. Particularly,

(i) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, then we have

$$\left|\frac{g(u)+g(v)}{2}-\frac{\Gamma(\mu+1)}{2(v-u)^\mu}\left[J^\mu_{u^+}g(v)+J^\mu_{v^-}g(u)\right]\right|$$
$$\leqslant\frac{v-u}{4}\left(|g'(u)|+\frac{1-\lambda}{\lambda}|g'(v)|\right)\left[\beta_{\frac{1}{2}}\left(\tfrac{1}{2},\mu+\tfrac{1}{2}\right)-\beta_{\frac{1}{2}}\left(\mu+\tfrac{1}{2},\tfrac{1}{2}\right)\right].$$

(ii) when $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then the inequality holds

$$\left|\frac{g(u)+g\left(u+e^{i\varphi}\eta(v,u)\right)}{2}-\frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^\mu_{u^+,k}g(u+e^{i\varphi}\eta(v,u))\right.\right.$$
$$\left.\left.+I^\mu_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\right|\leqslant\frac{v-u}{4}\left(|g'(u)|+\frac{1-\lambda}{\lambda}|g'(v)|\right)\left[\beta_{\frac{1}{2}}\left(\tfrac{1}{2},\tfrac{2\mu+k}{2k}\right)\right.$$
$$\left.-\beta_{\frac{1}{2}}\left(\frac{2\mu+k}{2k},\frac{1}{2}\right)\right].$$

Theorem 2.2. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on an open interval (u,v) such that $g'\in L[u,u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u)>0$ and $u<v$. If $|g'|^q, q>1$ is decreasing, measurable and λ_φ-preinvex on a closed interval $[u,v]$, then the inequality is true*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u))t+{}_{(u+e^{i\varphi}\eta(v,u))^-}I_\rho g(u)\right]\right|$$
$$\leqslant\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|^pd\zeta\right)^{\frac{1}{p}}\left(|g'(u)|^q+\left(\frac{1-\lambda}{\lambda}\right)|g'(v)|^q\right)^{\frac{1}{q}},\tag{2.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using equality (2.1), Hölder inequality and λ_φ-preinvexity of $|g'|^q$, we obtain

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.$$
$$\left.-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u))+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$=\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left|\int_0^1[\Upsilon(1-\zeta)-\Upsilon(\zeta)]g'(u+(1-\zeta)e^{i\varphi}\eta(v,u))d\zeta\right|$$
$$\leqslant\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|^pd\zeta\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1|g'(u+(1-\zeta)e^{i\varphi}\eta(v,u))|^qd\zeta\right)^{\frac{1}{q}}$$
$$\leqslant\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|^pd\zeta\right)^{\frac{1}{p}}$$
$$\times\left(|g'(u)|^q\int_0^1\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}d\zeta+|g'(v)|^q\int_0^1\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}\right)^{\frac{1}{q}}$$
$$=\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left(\int_0^1|\Upsilon(1)-\Upsilon(1-\zeta)-\Upsilon(\zeta)|^pd\zeta\right)^{\frac{1}{p}}\left(|g'(u)|^q\right.$$
$$\left.+\left(\frac{1-\lambda}{\lambda}\right)|g'(v)|^q\right)^{\frac{1}{q}}.$$

From which we obtain the desired inequality (2.5). □

Remark 2.3. Moreover, if the function ρ is specialized by

(i) $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, then our result, (2.5), becomes to the one (1.8).

(ii) $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$ with $\eta(v,u)=v-u$ and $\varphi=0$, the inequality holds

$$\left|\frac{g(u)+g(v)}{2}-\frac{\Gamma(\mu+1)}{2(v-u)^\mu}\left[J^\mu_{u^+}g(v)+J^\mu_{v^-}g(u)\right]\right|$$
$$\leqslant\frac{v-u}{2}\left\{\frac{\pi}{4}|g'(u)|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|g'(v)|^q\right\}^{\frac{1}{q}}\left(\frac{2-2^{1-\mu p}}{\mu p+1}\right)^{\frac{1}{p}}.$$

(iii) $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, we have

$$\left|\frac{g(u)+g\left(u+e^{i\varphi}\eta(v,u)\right)}{2}-\frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^\mu_{u^+,k}g(u+e^{i\varphi}\eta(v,u))\right.\right.$$
$$\left.\left.+I^\mu_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\right|$$
$$\leqslant\frac{e^{i\varphi}\eta(v,u)}{2}\left\{\frac{\pi}{4}|g'(u)|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|g'(v)|^q\right\}^{\frac{1}{q}}\left(\frac{2-2^{k-\mu p}}{\mu p+k}\right)^{\frac{1}{p}}.$$

Theorem 2.3. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on an open interval (u,v) such that $g'\in L[u,u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u)>0$ and $u<v$. If $|g'|^q$ is decreasing, measurable and λ_φ-preinvex on a closed interval $[u,v]$ for some fixed $q\geqslant 1$, then it holds*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.$$
$$\left.-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u))+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$\leqslant\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|d\zeta\right)^{1-\frac{1}{q}}(C|g'(u)|^q+D|g'(v)|^q)^{\frac{1}{q}},\tag{2.6}$$

where C and D are given by

$$C=\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}},\quad D$$
$$=\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}.$$

Proof. Employing (2.1), the power mean inequality and λ_φ-preinvexity of $|g'|^q$, we find

$$
\begin{aligned}
&\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.\\
&\quad\left.-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g\big(u+e^{i\varphi}\eta(v,u)\big)+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|\\
&\leqslant \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\big|g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\big|d\zeta\\
&\leqslant \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\big|g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\big|^q d\zeta\right)^{\frac{1}{q}}\\
&\leqslant \frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\\
&\quad\times\left(|g'(u)|^q\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}d\right.\\
&\quad\left.+|g'(v)|^q\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}\right)^{\frac{1}{q}}\\
&=\frac{e^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left(\int_0^1|\Upsilon(1-\zeta)-\Upsilon(\zeta)|d\zeta\right)^{1-\frac{1}{q}}\left(C|g'(u)|^q+D|g'(v)|^q\right)^{\frac{1}{q}}.
\end{aligned}
$$

This ends the proof of the theorem. □

Remark 2.4. Besides, we conclude that

(i) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, then inequality (2.6) reduces to the one (1.9).

(ii) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}, \eta(v,u)=v-u$ and $\varphi=0$, then we have

$$
\begin{aligned}
&\left|\frac{g(u)+g(v)}{2}-\frac{\Gamma(\mu+1))}{2(v-u)^\mu}\left[J^\mu_{u^+}g(v)+J^\mu_{v^-}g(u)\right]\right|\\
&\leqslant\frac{v-u}{2^{\frac{1}{q}}}\left(\frac{1-2^{-\mu}}{\mu+1}\right)^{\frac{q-1}{q}}\left(\frac{|g'(u)|^q}{2}\left[\beta_{\frac{1}{2}}\left(\frac{1}{2},\mu+\frac{1}{2}\right)\right.\right.\\
&\quad\left.-\beta_{\frac{1}{2}}\left(\mu+\frac{1}{2},\frac{1}{2}\right)\right]+\left(\frac{1-\lambda}{\lambda}\right)\frac{|g'(v)|^q}{2}\left[\beta_{\frac{1}{2}}\left(\frac{1}{2},\mu+\frac{1}{2}\right)\right.\\
&\quad\left.\left.-\beta_{\frac{1}{2}}\left(\mu+\frac{1}{2},\frac{1}{2}\right)\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

(iii) if $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then it holds

$$
\begin{aligned}
&\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.\\
&\quad\left.-\frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^\mu_{u^+,k}g\big(u+e^{i\varphi}\eta(v,u)\big)+I^\mu_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\right|\\
&\leqslant\frac{e^{i\varphi}\eta(v,u)}{2^{\frac{1}{q}}}\left(\frac{1-2^{\frac{-\mu}{k}}}{\mu+k}\right)^{\frac{q-1}{q}}\left(\frac{|g'(u)|^q}{2}\left[\beta_{\frac{1}{2}}\left(\frac{1}{2},\frac{2\mu+k}{2k}\right)\right.\right.\\
&\quad\left.-\beta_{\frac{1}{2}}\left(\frac{2\mu+k}{2k},\frac{1}{2}\right)\right]+\left(\frac{1-\lambda}{\lambda}\right)\frac{|g'(v)|^q}{2}\left[\beta_{\frac{1}{2}}\left(\frac{1}{2},\frac{2\mu+k}{2k}\right)\right.\\
&\quad\left.\left.-\beta_{\frac{1}{2}}\left(\frac{2\mu+k}{2k},\frac{1}{2}\right)\right]\right)^{\frac{1}{q}}.
\end{aligned}
$$

3. Trapezoidal type inequalities for twice differential functions

In this section, we are interesting in the study of the trapezoidal type inequalities for λ_φ–preinvex functions, which are twice differentiable. To obtain these inequalities, we first verify the following lemma.

Lemma 3.1. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable functions on an open interval (u,v) such that $g''\in L[u,u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u)>0$ and $u<v$. Then, it holds*

$$
\begin{aligned}
&\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\\
&\quad-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g\big(u+e^{i\varphi}\eta(v,u)\big)+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\\
&=\frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left[\int_0^1[\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]g''\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\right],
\end{aligned}
\tag{3.1}
$$

where

$$
\Omega(\zeta)=\int_0^\zeta\Upsilon(s)ds,\quad \Upsilon(s)=\int_0^s\frac{\rho(e^{i\varphi}\eta(v,u)x)}{x}dx<\infty. \tag{3.2}
$$

Proof. It follows from the integrating by parts that

$$
\begin{aligned}
\hbar&=\int_0^1[\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]g''\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\\
&=\frac{-1}{e^{i\varphi}\eta(v,u)}\int_0^1[\Upsilon(\zeta)-\Upsilon(1-\zeta)]g'\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\\
&=\frac{1}{(e^{i\varphi}\eta(v,u))^2}\left[\big\{g(u)+g\big(u+e^{i\varphi}\eta(v,u)\big)\big\}\Upsilon(1)\right.\\
&\quad+\int_0^1\rho(e^{i\varphi}\eta(v,u)t)g\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\\
&\quad\left.-\int_0^1\rho(e^{i\varphi}\eta(v,u)(1-\zeta))g\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\right].
\end{aligned}
$$

By making use of the substitution $x=u+(1-\zeta)e^{i\varphi}\eta(v,u)$, it yields

$$
\begin{aligned}
\hbar&=\frac{1}{(e^{i\varphi}\eta(v,u))^2}\left[\big\{g(u)+g\big(u+e^{i\varphi}\eta(v,u)\big)\big\}\Upsilon(1)\right.\\
&\quad\left.-\int_u^{u+e^{i\varphi}\eta(v,u)}\frac{\rho(x-u)}{x-u}dx-\int_u^{u+e^{i\varphi}\eta(v,u)}\frac{\rho(u+e^{i\varphi}\eta(v,u)-x)}{u+e^{i\varphi}\eta(v,u)-x}dx\right]\\
&=\frac{1}{(e^{i\varphi}\eta(v,u))^2}\left[\big\{g(u)+g\big(u+e^{i\varphi}\eta(v,u)\big)\big\}\Upsilon(1)-\left({}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)+{}_{u^+}I_\rho g\big(u+e^{i\varphi}\eta(v,u)\big)\right)\right].
\end{aligned}
\tag{3.3}
$$

Then, A simple calculation gives the equality (3.2). □

Remark 3.1. Particularly, it holds that

(i) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, then the identity (3.1) reduces to the one (1.10).

(ii) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}, \eta(v,u)=v-u$ and $\varphi=0$, then identity (3.1) reduces to the following result

$$
\begin{aligned}
&\frac{g(u)+g(v)}{2}-\frac{\Gamma(\mu+1))}{2(v-u)^\mu}\left[I^\mu_{u^+}g(v)+I^\mu_{v^-}g(u)\right]\\
&\quad=\frac{(v-u)^2}{2(\mu+1)}\int_0^1\left[1-(1-\zeta)^{\mu+1}-\zeta^{\mu+1}\right]g''(\zeta u+(1-\zeta)v)d\zeta
\end{aligned}
$$

which was obtained by Wang et al. [38].

(iii) if $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then we have

$$\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^{\mu}_{u^+,k}g(u+e^{i\varphi}\eta(v,u)) + I^{\mu}_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]$$
$$= \frac{(e^{i\varphi}\eta(v,u))^2}{2(\mu+k)} \times \int_0^1 \left[1-(1-\zeta)^{\frac{\mu}{k}+1} - \zeta^{\frac{\mu}{k}+1}\right] g''(u+(1-\zeta)e^{i\varphi}\eta(v,u))d\zeta.$$

Theorem 3.1. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on an open interval (u,v) such that $g''\in L[u,u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u)>0$ and $u<v$. If $|g''|$ is decreasing, measurable and λ_φ-preinvex on a closed interval $[u,v]$, then we have*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right| \leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}(A|g''(u)|+B|g''(v)|), \tag{3.4}$$

where the constants A, and B are defined by

$$A=\int_0^1 \frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|d\zeta,$$
$$B=\int_0^1 \frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|d\zeta.$$

Proof. Invoking the result (3.1) and the property of modulus, it finds

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$= \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left|\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]g''(u+(1-\zeta)e^{i\varphi}\eta(v,u))d\zeta\right|$$
$$\leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\int_0^1 |\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)||g''(u+(1-\zeta)e^{i\varphi}\eta(v,u))|d\zeta.$$

Notice that $|g''|$ is λ_φ-preinvex function on interval $[u,v]$, for any $\zeta\in(0,1)$, we conclude

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$\leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left[\int_0^1 |\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\left(\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}|g''(u)|d\zeta + \frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}|g''(v)|\right)\right]$$
$$\leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left[|g''(u)|\int_0^1 |\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}d\zeta + |g''(v)|\int_0^1 |\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}\right]$$
$$= \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}(A|g''(u)|+B|g''(v)|).$$

This completes the proof of the theorem. □

Remark 3.2. Indeed, we have the following special cases:

(i) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, then inequality (3.4) reduces to the one (1.11).

(ii) if $\rho(\zeta)=\frac{\zeta^\mu}{\Gamma(\mu)}$, $\eta(v,u)=v-u$ and $\varphi=0$, then we have

$$\left|\frac{g(u)+g(v)}{2} - \frac{\Gamma(\mu+1))}{2(v-u)^\mu}\left[J^\mu_{u^+}g(v)+J^\mu_{v^-}g(u)\right]\right| \leqslant \frac{(v-u)^2}{4(\mu+1)}\times\left\{|g''(u)|\left[\frac{\pi}{2}-\beta\left(\frac{3}{2},\mu+\frac{3}{2}\right)-\beta\left(\mu+\frac{5}{2},\frac{1}{2}\right)\right] + |g''(v)|\left[\frac{\pi}{2}-\beta\left(\frac{1}{2},\mu+\frac{5}{2}\right)-\beta\left(\mu+\frac{3}{2},\frac{3}{2}\right)\right]\right\}.$$

(iii) if $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then it hods

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^\mu_{u^+,k}g(u+e^{i\varphi}\eta(v,u)) + I^\mu_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\right|$$
$$\leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{4(\mu+k)}\left\{|g''(u)|\left[\frac{\pi}{2}-\beta\left(\frac{3}{2},\frac{2\mu+3k}{2k}\right)-\beta\left(\frac{2\mu+5k}{2k},\frac{1}{2}\right)\right] + |g''(v)|\left[\frac{\pi}{2}-\beta\left(\frac{1}{2},\frac{2\mu+5k}{2k}\right)-\beta\left(\frac{2\mu+3k}{2k},\frac{3}{2}\right)\right]\right\}.$$

Theorem 3.2. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on an open interval (u,v) such that $g''\in L[u,u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u)>0$ and $u<v$. If $|g''|^q$ is decreasing, measurable and λ_φ-preinvex on a closed interval $[u,v]$, then*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right| \leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|^p d\zeta\right)^{\frac{1}{p}}$$

where $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Applying equality (3.1), Hölder inequality and λ_φ-preinvexity of $|g''|^q$, implies

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho g(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$= \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left|\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]g''(u+(1-\zeta)e^{i\varphi}\eta(v,u))d\zeta\right|$$
$$\leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|^p d\zeta\right)^{\frac{1}{p}} \times\left(\int_0^1|g''(u+(1-\zeta)e^{i\varphi}\eta(v,u))|^q d\zeta\right)^{\frac{1}{q}} \leqslant \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}$$
$$\times\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|^p d\zeta\right)^{\frac{1}{p}} \times\left(|g''(u)|^q\int_0^1\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}d\zeta + |g''(v)|^q\int_0^1\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}\right)^{\frac{1}{q}}$$
$$= \frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\frac{\pi}{4}\right)^{\frac{1}{q}}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|^p d\zeta\right)^{\frac{1}{p}} \times\left(|g''(u)|^q+\left(\frac{1-\lambda}{\lambda}\right)|g''(v)|^q\right)^{\frac{1}{q}}.$$

This completes the proof of the theorem. □

Remark 3.3. However, if

(i)$\rho(\zeta)=\frac{\zeta^{\mu}}{\Gamma(\mu)}$, then inequality (3.5) reduces to the one (1.12).

(ii)$\rho(\zeta)=\frac{\zeta^{\mu}}{\Gamma(\mu)}$ with $\eta(v,u)=v-u$ and $\varphi=0$, then

$$\left|\frac{g(u)+g(v)}{2}-\frac{\Gamma(\mu+1))}{2(v-u)^{\mu}}\left[J^{\mu}_{u^+}g(v)+J^{\mu}_{v^-}g(u)\right]\right|$$
$$\leqslant\frac{(v-u)^2}{2(\mu+1)}\left(1-2^{1-\mu}\right)\left\{\frac{\pi}{4}|g''(u)|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|g''(v)|^q\right\}^{\frac{1}{q}}.$$

(iii) $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}>-\frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\right.$$
$$\left.\times\left[I^{\mu}_{u^+,k}g\big(u+e^{i\varphi}\eta(v,u)\big)+I^{\mu}_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\right|$$
$$\leqslant\frac{\left(e^{i\varphi}\eta(v,u)\right)^2}{2(\mu+k)}\left(1-2^{\frac{k-\mu}{k}}\right)\left\{\frac{\pi}{4}|g''(u)|^q+\frac{\pi}{4}\left(\frac{1-\lambda}{\lambda}\right)|g''(v)|^q\right\}^{\frac{1}{q}}.$$

Theorem 3.3. *Let $g:[u,v]\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on an open interval (u,v) such that $g''\in L[u,u+e^{i\varphi}\eta(v,u)]$ with $\eta(v,u)>0$ and $u<v$. If $|g''|^q$ is decreasing, measurable and λ_{φ}-preinvex on a closed interval $[u,v]$ for some $q\geqslant 1$, then*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.$$
$$\left.-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_{\rho}g\big(u+e^{i\varphi}\eta(v,u)\big)>+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|d\zeta\right)^{1-\frac{1}{q}}$$
$$\times\left(C|g''(u)|^q+D|g''(v)|^q\right)^{\frac{1}{q}},\tag{3.6}$$

where C and D are given by

$C=\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\frac{\sqrt{\zeta}}{2\sqrt{1-\zeta}}$,

$D=\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}$.

Proof. *It follows from equality (3.1), the power mean inequality and λ_{φ}-preinvexity of $|g''|^q$ that*

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.$$
$$\left.-\frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_{\rho}g\big(u+e^{i\varphi}\eta(v,u)\big)+{}_{(u+e^{i\varphi}\eta(v,u))^-}\rho g(u)\right]\right|$$
$$\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\left|g''\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\right|d\zeta$$
$$\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|d\zeta\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\left|g''\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\right|^q d\zeta\right)^{\frac{1}{q}}$$
$$\leqslant\frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|d\zeta\right)^{1-\frac{1}{q}}$$
$$\times\left(|g''(u)|^q\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\frac{\sqrt{\zeta}}{2\lambda\sqrt{1-\zeta}}d\zeta\right.$$
$$\left.+|g''(v)|^q\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|\frac{(1-\lambda)\sqrt{1-\zeta}}{2\lambda\sqrt{\zeta}}\right)^{\frac{1}{q}}$$
$$=\frac{(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left(\int_0^1|\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)|d\zeta\right)^{1-\frac{1}{q}}$$
$$\times(C|g''(u)|^q+D|g''(v)|^q)^{\frac{1}{q}}.$$

This means that (3.6) is valid. □

Remark 3.4. It is obvious that

(i) if $\rho(\zeta)=\frac{\zeta^{\mu}}{\Gamma(\mu)}$, then inequality (3.6) reduces to the one (1.13).

(ii) if $\rho(\zeta)=\frac{\zeta^{\mu}}{\Gamma(\mu)}$ with $\eta(v,u)=v-u$ and $\varphi=0$, then

$$\left|\frac{g(u)+g(v)}{2}-\frac{\Gamma(\mu+1))}{2(v-u)^{\mu}}\left[J^{\mu}_{u^+}g(v)+J^{\mu}_{v^-}g(u)\right]\right|$$
$$\leqslant\frac{(v-u)^2}{2(\mu+1)}\left(1-2^{1-\mu}\right)^{\frac{q-1}{q}}\left(\frac{|g''(u)|^q}{2}\left[\frac{\pi}{2}-\beta\left(\frac{3}{2},\mu+\frac{3}{2}\right)\right.\right.$$
$$\left.\left.-\beta\left(\mu+\frac{5}{2},\frac{1}{2}\right)\right]+\frac{|g''(v)|^q}{2}\left[\frac{\pi}{2}-\beta\left(\frac{1}{2},\mu+\frac{5}{2}\right)-\beta\left(\mu+\frac{3}{2},\frac{3}{2}\right)\right]\right)^{\frac{1}{q}}.$$

(iii) if $\rho(\zeta)=\frac{\zeta^{\frac{\mu}{k}}}{k\Gamma_k(\mu)}$, then

$$\left|\frac{g(u)+g(u+e^{i\varphi}\eta(v,u))}{2}\right.$$
$$\left.-\frac{\Gamma_k(\mu+k))}{2(e^{i\varphi}\eta(v,u))^{\frac{\mu}{k}}}\left[I^{\mu}_{u^+,k}g\big(u+e^{i\varphi}\eta(v,u)\big)+I^{\mu}_{(u+e^{i\varphi}\eta(v,u))^-,k}g(u)\right]\right|$$
$$\leqslant\frac{\left(e^{i\varphi}\eta(v,u)\right)^2}{2(\mu+k)}\left(1-2^{\frac{k-\mu}{k}}\right)^{\frac{q-1}{q}}\left(\frac{|g''(u)|^q}{2}\left[\frac{\pi}{2}-\beta\left(\frac{3}{2},\frac{2\mu+3k}{2k}\right)\right.\right.$$
$$\left.-\beta\left(\frac{2\mu+5k}{2k},\frac{1}{2}\right)\right]\frac{|g''(v)|^q}{2}\left[\frac{\pi}{2}-\beta\left(\frac{1}{2},\frac{2\mu+5k}{2k}\right)\right.$$
$$\left.\left.-\beta\left(\frac{2\mu+3k}{2k},\frac{3}{2}\right)\right]\right)^{\frac{1}{q}}.$$

4. Applications for the moments

Distribution functions and density functions provide complete descriptions of the distribution of probability for a given random variable. However, they do not allow us to easily make comparisons between two different distributions. The set of moments that uniquely characterizes the distribution under reasonable conditions are useful in making comparisons. Knowing the probability function, we can determine moments if they exist. Applying the mathematical inequalities, some estimations for the moments of random variables were recently studied by many authors, for more details see [1,3,4,16–18,35,7].

Let χ be a random variable whose probability function is $g:\Im\subset\mathbb{R}\to\mathbb{R}_+$, where g is λ_{φ}-preinvex function on the interval of real numbers $\Im$ such that $u,v\in\Im$ with $u<v$. Denote by $M_r(x)$ the r^{th} moment about any arbitrary point x of the random variable χ, $r\geqslant 0$, defined as

$$M_r(x)=\int_u^{u+e^{i\varphi}}(z-x)^r g(z)dz,\quad r=0,1,2,\ldots\tag{4.1}$$

In view of (4.1), we obtain the following propositions:

Proposition 4.1. *Let $r\geqslant 1$ and χ be a random variable with probability function $g:\Im\subset\mathbb{R}\to\mathbb{R}_+$, where g is a λ_{φ}-preinvex function on the interval of real numbers $\Im$ such that $u,v\in\Im$ with $u<v$. Then, it holds*

$$\frac{M_r(u)+M_r(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho M_r\big(u+e^{i\varphi}\eta(v,u)\big) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho M_r(u)\right]$$
$$= \frac{re^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left[\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)]M_{r-1}\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\right]$$
$$= \frac{re^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left\{\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)] \times \left(\int_u^{u+e^{i\varphi}} \big(z-\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\big)^{r-1}g(z)dz\right)d\zeta\right\}.$$

Proof. It is obtained directly by using equality (2.1) with $g(x)=M_r(x)$. □

Proposition 4.2. *Let $r \geqslant 1$ and χ be a random variable with probability function $g:\mathfrak{I}\subset\mathbb{R}\to\mathbb{R}_+$, where g is a λ_φ-preinvex function on the interval of real numbers $\mathfrak{I}$ such that $u,v\in\mathfrak{I}$ with $u<v$. If the function $|g|$ is bounded, then*

$$\left|\frac{M_r(u)+M_r\big(u+e^{i\varphi}\eta(v,u)\big)}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho M_r(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho M_r(u)\right]\right|$$
$$\leqslant \frac{\big(e^{i\varphi}\eta(v,u)\big)^r\|g\|_\infty}{2\Upsilon(1)}\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)]((1-\zeta)^r-\zeta^r)d\zeta.$$

Proof. Employing Proposition 4.1 and the boundedness of $|g|$ deduces

$$\left|\frac{M_r(u)+M_r(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho M_r\big(u+e^{i\varphi}\eta(v,u)\big) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho M_r(u)\right]\right|$$
$$\leqslant \frac{re^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left\{\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)] \times \left(\int_u^{u+e^{i\varphi}} \left|z-\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\right|^{r-1}|g(z)|dz\right)d\zeta\right\}$$
$$\leqslant \frac{(e^{i\varphi}\eta(v,u))^r\|g\|_\infty}{2\Upsilon(1)}\int_0^1 [\Upsilon(1-\zeta)-\Upsilon(\zeta)]((1-\zeta)^r-\zeta^r)d\zeta.$$

This completes the proof of the proposition. □

Proposition 4.3. *Let $r \geqslant 2$ and χ be a random variable with probability function $g:\mathfrak{I}\subset\mathbb{R}\to\mathbb{R}_+$, where g is a λ_φ-preinvex function on the interval of real numbers $\mathfrak{I}$ such that $u,v\in\mathfrak{I}$ with $u<v$. Then, we have*

$$\frac{M_r(u)+M_r(u+e^{i\varphi}\eta(v,u))}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho M_r\big(u+e^{i\varphi}\eta(v,u)\big) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho M_r(u)\right]$$
$$= \frac{r(r-1)(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)} \times \left[\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]M_{r-2}\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)d\zeta\right]$$
$$= \frac{r(r-1)(e^{i\varphi}\eta(v,u))^2}{2\Upsilon(1)}\left\{\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)] \times \left(\int_u^{u+e^{i\varphi}} \big(z-\big(u+(1-\zeta)e^{i\varphi}\eta(v,u)\big)\big)^{r-2}g(z)dz\right)d\zeta\right\}.$$

Proof. The desired result is a direct consequence of equality (3.1) for $g(x)=M_r(x)$. □

Proposition 4.4. *Let $r \geqslant 2$ and χ be a random variable with probability function $g:\mathfrak{I}\subset\mathbb{R}\to\mathbb{R}_+$, where g is a λ_φ-preinvex function on the interval of real numbers $\mathfrak{I}$ such that $u,v\in\mathfrak{I}$ with $u<v$. If the function $|g|$ is bounded, then*

$$\left|\frac{M_r(u)+M_r\big(u+e^{i\varphi}\eta(v,u)\big)}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho M_r(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho M_r(u)\right]\right|$$
$$\leqslant \frac{r\big(e^{i\varphi}\eta(v,u)\big)^r\|g\|_\infty}{2\Upsilon(1)}\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]\Big((1-\zeta)^{r-1}-\zeta^{r-1}\Big)d\zeta.$$

Proof. Employing Proposition 4.3 and the boundedness of $|g|$ implies

$$\left|\frac{M_r(u)+M_r\big(u+e^{i\varphi}\eta(v,u)\big)}{2} - \frac{1}{2\Upsilon(1)}\left[{}_{u^+}I_\rho M_r(u+e^{i\varphi}\eta(v,u)) + {}_{(u+e^{i\varphi}\eta(v,u))^-}\rho M_r(u)\right]\right|$$
$$\leqslant \frac{re^{i\varphi}\eta(v,u)}{2\Upsilon(1)}\left\{\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)] \left(\int_u^{u+e^{i\varphi}} |z-(u+(1-\zeta)e^{i\varphi}\eta(v,u))|^{r-2}|g(z)|dz\right)d\zeta\right\};$$
$$\leqslant \frac{\big(e^{i\varphi}\eta(v,u)\big)^r\|g\|_\infty}{2\Upsilon(1)}\int_0^1 [\Omega(1)-\Omega(\zeta)-\Omega(1-\zeta)]\Big((1-\zeta)^{r-1}-\zeta^{r-1}\Big)d\zeta.$$

This completes the proof of the proposition. □

5. Conclusion

In this paper, we established the trapezoidal type inequalities involving generalized fractional integrals for the λ_φ–preinvex functions, which are differentiable or twice differentiable. Furthermore, we obtained some new inequalities of trapezoidal-type for differentiable λ_φ–preinvex functions by applying classical and Riemann–Liouville fractional integrals. The results presented in this paper would provide generalizations of those given in such earlier works [8,41,39]. In the future, you may apply the generalized fractional integrals to various models of fractional calculus such as Atangana–Baleanu and Prabhakar models.

Declaration of Competing Interest

The authors declare no conflict of interest.

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